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Canonical Realizations of the Lie Algebra $sp(2n, R)$

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Abstract

The generators of the Lie algebra of the symplectic group $sp(2n, R)$ are, recurrently, realized by means of polynomials in the quantum canonical variables p_i and q_i. These realizations are skew-Hermitian, the Casimir operators are realized by constant multiples of identity elements, and, depending on the number of the canonical pairs used, they depend on d, $d = 1, 2, \ldots, n$ free real parameters.

1. *Introduction*

The object of this paper is to present a large class of realizations of the Lie algebra of the real symplectic group in the Weyl algebra, viz., through polynomials in quantum canonical variables q_i, p_i with various good properties.

For physical relevance of canonical realizations of Lie algebras in general we refer to the review articles Cordero and Ghirardi: (1972) and Wolf (1975) and the references therein. As to the symplectic group, we remember only that it occurs in physics as a subgroup of general canonical transformations, namely, of the group $ISP(2n, R)$ of inhomogeneous linear transformations which leave the commutation relations of n canonical pairs $[p_i, q_j] = \delta_{ij}z$, $[q_i, q_j] = [p_i, p_j] = 0, i, j = 1, 2, \ldots, n$ invariant (Wolf, 1975; Moshinsky, 1973). The Lie algebra $sp(2n, R)$ is the dynamical algebra of the n-dimensional harmonical oscillator (Cordero and Ghirardi, 1972).

The proposed canonical realizations have common features with those of real forms of the other classical Lie algebras A_n, B_n, D_n presented in Havlíček and Lassner (1976) and Havlícek and Exner (1975a). The realizations are recurrently defined by means of $2n - 1$ canonical pairs and a canonical realization of the algebra $sp(2n - 2, R)$ with one free real parameter. Using, for realization of the auxiliary Lie algebra $sp(2n - 2, R)$, either the trivial one or the realization defined by the same formulas, etc., we obtain a set of realizations of $sp(2n, R)$.

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Realizations of this set are in one-to-one correspondence with the sequences $(d; 0, \ldots, 0, \alpha_{n-d+1}, \ldots, \alpha_n)$, $d = 1, 2, \ldots, n$, $\alpha_i \in R$; these sequences we call signatures. The generators of $sp(2n, R)$ in a realization with signature $(d; 0, \ldots, 0, \alpha_{n-d+1}, \ldots, \alpha_n)$ lie in the Weyl algebra $W_{2N(d)}$ where $N(d) = d(2n - d)$, i.e., they are polynomials in $N(d)$ canonical pairs. All realizations are Schur realizations, which means that every Casimir operator is realized by a complex multiple of the identity element and all realizations are skew-hermitian with respect to an involution defined on the Weyl algebra. Two realizations characterized by different signatures cannot be transformed from one to another by means of endomorphisms of the Weyl algebra.

The number $N(d) = d\,(2n - d)$ of pairs used in the construction of the realizations with signature $(d; 0, \ldots, 0, \alpha_{n-d+1}, \ldots, \alpha_n)$ is smallest for $d = 1$ when $N(1) = 2n - 1$. Of course, this is not the minimal number of canonical pairs that allows a faithful realization of $sp(2n, R)$. The well-known minimal realization τ_1 of $sp(2n, R)$ is given by the following expressions:

$$q_i p_j + \tfrac{1}{2}\delta_{ij}, \qquad iq_iq_j, \qquad ip_ip_j, \qquad i,j = 1, \ldots, n \tag{1.1}$$

where n canonical pairs are used. On the basis of the result of Joseph (1974, Lemma 1) it could be proved for $n \geqslant 2$ that in any realization τ of $sp(2n, R)$ in the quotient division ring $D_{2(2n-2)}$ of $W_{2(2n-2)}$ (i.e., by means of rational functions in $2n - 2$ canonical pairs)

$$\tau(z) = \tau_1(z) = \lambda_z \mathbb{1}, \qquad \lambda_z \in C$$

holds for any Casimir operator z of $sp(2n, R)$ (Havlíček and Lassner, 1976a).

So, the possibility of obtaining realizations of $sp(2n, R)$ in which Casimir operators are realized by expressions other than in realization τ_1 would appear only in W_{2N} or D_{2N} with $N \geqslant 2n - 1$. The mentioned one-parameter set of realizations with signatures $(1; 0, \ldots, 0, \alpha_n)$ in $W_{2(2n-1)}$ shows that N equals just $2n - 1$ and that canonical realizations are given by polynomials. Further, in these realizations, e.g., the quadratic Casimir operator $C^{(2)}$ depends on the parameter α_n, $\tau(C^{(2)}) = -2(\alpha_n{}^2 + n^2)$, whereas for the realization τ_1 one finds $\tau_1(C^{(2)}) = -n^2 - \frac{1}{2}n$.

The fact that these realizations are still Schur realizations is not accidental, since it could be proved that in $W_{2(2n-1)}$ any realization of $sp(2n, R)$ is a Schur realization (Havlíček and Lassner, 1976a).

In Section 4 we show how this "minimal" one-parameter set of realization of $sp(2n, R)$ can be obtained by means of the one-parameter set of minimal realizations of $gl(2n, R)$ given in our paper (Havlíček and Lassner, 1976). We discuss a formula very useful for constructing canonical realizations of any finite-dimensional Lie algebra.

Some considerations determine, for any compact classical Lie algebra, the minimal number of canonical pairs needed for skew-Hermitian realizations.

2. *Preliminaries*

In the Lie algebra of the symplectic group, i.e., the group of linear transformations of the $2n$-dimensional vector space that left invariant the bilinear form

$$\sum_{i=1}^{n} (x^i y^{-i} - x^{-i} y^i) \tag{2.1}$$

we choose a basic consisting of $n(2n + 1)$ generators $X_\beta{}^\alpha = -\epsilon_\alpha\epsilon_\beta X_{-\alpha}^{-\beta}$ $\alpha, \beta = -n, \ldots, -1, 1, \ldots, n$ satisfying the commutation rules

$$[X_\beta^\alpha, X_\delta{}^\gamma] = \delta_\beta{}^\gamma X_\delta{}^\alpha - \delta_\delta{}^\alpha X_\beta{}^\gamma + \epsilon_\alpha\epsilon_\beta\delta_{\bar\delta}^{-\beta} X_{-\alpha}^\gamma + \epsilon_\beta\epsilon_\gamma\delta_{-\alpha}^\gamma X_{\bar\delta}^{-\beta},$$

$$\epsilon_\alpha = \operatorname{sgn}\alpha \tag{2.2}$$

A $2n \times 2n$ matrix representation is

$$(X_\beta{}^\alpha)_{\gamma\delta} = \delta_{\gamma\alpha}\delta_{\beta\delta} - \epsilon_\alpha\epsilon_\beta\delta_{-\alpha\delta}\delta_{-\beta\gamma} \tag{2.3}$$

A *canonical realization* of a Lie algebra L is a homomorphism of L in the *Weyl algebra* W_{2N}, the associative algebra over C with identity generated by $2N$ elements $q_i, p_i, i = 1, 2, \ldots, N$ with commutation relations

$$[p_i, q_j] = \delta_{ij}\mathbb{1}$$

The homomorphism τ extends naturally to a homomorphism (denoted by the same symbol τ) of the enveloping algebra UL of L into W_{2N}.

If in a realization of L every Casimir operator, i.e., every element from the center of the enveloping algebra of L, is realized by a multiple of the identity element, then the realization is called a *Schur realization.* Two realizations τ and τ' of L in W_{2N} are called *related* if an endomorphism ϑ of W_{2N} exists such that either $\vartheta \cdot \tau = \tau'$ or $\vartheta \cdot \tau' = \tau$. In W_{2N} we define an *involution* induced by

$$p_i^+ = -p_i, \qquad q_i^+ = q_i \tag{2.4}$$

A canonical realization τ of the real Lie algebra L is called *skew-Hermitian* iff $\tau(x)^+ = -\tau(x)$ for all $x \in L$.

3. *Canonical Realization of* $sp(2n, R)$

Theorem 1. Let $Z^i{}_j$ be a canonical realization of $sp(2n - 2, R)$ in W_{2m}. Then the generators

$$\begin{aligned}
X_j^i &= q_i p_j - \epsilon_i\epsilon_j q_{-j}p_{-i} + Z_j^i \\
X_n^j &= q_j(q \cdot p + n - i\alpha) - \epsilon_j q_0 p_{-j} + Z_k^j q_k \\
X_j^n &= -p_j - \epsilon_j q_{-j} p_0 \\
X_{-n}^n &= -2p_0 \\
X_n^{-n} &= 2q_0(q \cdot p + n - i\alpha) + \epsilon_l Z_{\bar k}^{-l}\, q_l q_k \\
X_n^n &= -q_0 p_0 - q \cdot p - (n - i\alpha) \cdot \mathbb{1}
\end{aligned} \tag{3.1}$$

$$i, j, k, l = -(n-1), \ldots, -1, 1, \ldots, n-1$$

where $\alpha \in \mathbb{C}$ and $q \cdot p = q_0 p_0 + q_k p_k$, define a realization of $sp(2n,R)$ in $W_{2(2n-1+m)}$. This realization has the following properties:

(i) The realization is skew-Hermitian if α is real and if $Z^i{}_j$ is skew-Hermitian.
(ii) The realization is a Schur realization if $Z^i{}_j$ is a Schur realization.
(iii) Two realizations (3.1) with different parameters are nonrelated.
(iv) Two realizations (3.1) differing only in the realizations of $sp(2n-2,R)$ are related if and only if these realizations of $sp(2n-2,R)$ are related.

Proof. The verification that the generators (3.1) fulfil the commutation relations (2.2) of $sp(2n,R)$ and that they are skew-Hermitian under the involution defined by (2.4) is straightforward and will be omitted here. In the proof of (ii)–(iv) we use two assertions that are easily provable using the relation $[qp, q^k p^s] = (k-s)q^k p^s$ for each canonical pair occuring in W_{2N}.

Assertion 1. If $x \in W_{2N}$ commutes with p_i (or, respectively, q_i) then x does not depend on q_i (p_i).

Assertion 2. Assume that for $x \in W_{2N}$ there holds

$$[q_1 p_1 + \cdots + q_{N'} p_{N'}, x] = m \,.\, x$$

for some $m = 0, \pm 1, \pm 2, \ldots$ where $N' \leqslant N$.
Then

$$x = \sum_{\substack{k,l \\ k-l \equiv m}} \alpha_{kl} \cdot q^k p^l$$

where

$$\alpha_{kl} q^k p^l \equiv \alpha_{k_1 \ldots k_{N'} l_1 \ldots l_{N'}} \cdot q_1^{k_1} \cdots q_{N'}^{k_{N'}} p_1^{l_1} \cdots p_{N'}^{l_{N'}}$$
$$k - l \equiv k_1 + \cdots + k_{N'} - l_1 - \cdots - l_{N'}$$

and α_{kl} do not depend on $q_1, \ldots, q_{N'}, p_1, \ldots, p_{N'}$

(ii) Let Y be an element from the center of the enveloping algebra of $sp(2n,R)$ in its realization induced by (3.1). By definition Y commutes with all generators of $sp(2n,R)$. First, we consider the consequence of this fact using only generators that do not depend on $Z^i{}_j$:

$$[Y, X^n_{-n}] = 0, \text{ i.e., } [Y, p_0] = 0 \tag{3.2}$$

$$[Y, X^n_{-j}] = 0, \text{ i.e., } [Y, p_j] + \epsilon_j [Y, q_{-j} p_0] = 0 \tag{3.3}$$

$$[Y, X^n_n] = 0, \text{ i.e., } [Y, 2q_0 p_0 + q_j p_j] = 0 \tag{3.4}$$

From (3.2) it follows, owing to Assertion 1, that Y does not depend on q_0. Therefore Y can be written in the form

$$Y = \sum \gamma_r p_0^r \tag{3.5}$$

with

$$\gamma_r = \sum_{k,l} \alpha^r_{kl} q^k p^l$$

where α^r_{kl} are polynomials in $Z^i{}_j$. We show that only α^0_{00} can differ from zero. For γ_r relation (3.4) gives

$$[q_j p_j, \gamma_r] = 2r\gamma_r, \qquad r = 0, 1, \ldots \tag{3.4'}$$

Taking (3.3) for the zeroth power in p_0 we obtain further

$$[\gamma_0, p_j] = 0, \qquad j = -(n-1), \ldots, -1, 1, \ldots, n-1$$

and, owing to Assertion 1, γ_0 does not depend on q_i:

$$\gamma_0 = \sum_l \alpha^0_{0l} p^l$$

Equation (3.4′) in combination with Assertion 2 leads immediately to

$$\gamma_0 = \alpha^0_{00}(Z^i_j)$$

since the condition $k - l = -l_{-(n-1)} - \cdots - l_{n-1} = 0$ necessary for α^0_{0l} to be nonzero is fulfilled only for $l_{-(n-1)} = \cdots = l_{n-1} = 0$. If we take (3.3) for the first power in p_0 we obtain

$$[\gamma_1, p_j] + \epsilon_j [\gamma_0, q_{-j}] = 0$$

which, due to Assertion 1, implies

$$\gamma_1 = \sum_l \alpha^1_{0l} p^l \tag{3.5'}$$

Equation (3.4′) and Assertion 2 give, for the difference $k - l = -l_{-(n-1)} - \ldots - l_{n-1}$ the condition $l_{-(n-1)} + \ldots + l_{n-1} = -2$ which cannot be fulfilled for non-negative integers l's. Consequently all the coefficients α^1_{0l} in (3.5′) must be zero, i.e.,

$$\gamma_1 = 0$$

Putting γ_1 in (3.3) taken for the second power in p_0 we get by the same arguments $\gamma_2 = 0$, and so on. Thus we can show that $Y = \alpha^0_{00}$ is a polynomial in the generators $Z^i{}_j$ of $sp(2n-2, R)$ only. Hence from the condition that Y commutes with the remaining three types of generators $X^i{}_j, X^i{}_n, X^{-n}{}_n$, which contain the generators $Z^i{}_j$ it follows that $Y \equiv \lambda \cdot \mathbb{1}, \lambda \in \mathbb{C}$ if $Z^i{}_j$ form a Schur-realization and (ii) is proved.

In the proof of (iii) and (iv) we use very similar arguments. Let ϑ be an endomorphism of the Weyl algebra $W_{2(2n-1+m)}$ which connects two realizations (3.1)

$$\vartheta(\tilde{X}^\alpha_\beta) = X^\alpha_\beta; \qquad \alpha, \beta = -n, \ldots, -1, 1, \ldots, n \tag{3.6}$$

where the realization $\tilde{X}^\alpha_\beta$ depends on $\tilde{\alpha}$ and $\tilde{Z}^i_j$ and the realization X^α_β depends on α and Z^i_j. From the equations (3.6) we use first only three types

$$\vartheta(\tilde{X}^n{}_{-n}) = X^n{}_{-n}, \text{ i.e., } \vartheta(p_0) = p_0 \tag{3.7}$$

$$\vartheta(\tilde{X}_j^n) = X_j^n, \text{ i.e., } \vartheta(p_j + \epsilon_j q_{-j} p_0) = p_j + \epsilon_j q_{-j} p_0 \tag{3.8}$$

$$\vartheta(\tilde{X}_n^n) = X_n^n, \text{ i.e., } \vartheta(q_0 p_0 + q \cdot p + \tilde{\alpha}\mathbb{1}) = q_0 p_0 + q \cdot p + \alpha \cdot \mathbb{1} \tag{3.9}$$

Now we turn to $\vartheta(q_i)$ and determine it from its commutation relations with (3.7)-(3.9). Because of (3.7)

$$[\vartheta(q_i) - q_i, p_0] = 0$$

holds which, owing to Assertion 1, implies that $[\vartheta(q_i) - q_i]$ does not depend on q_0. We denote this element again by Y and can write

$$Y = \vartheta(q_i) - q_i = \sum_r \gamma_r \cdot p_0^r$$

with

$$\gamma_r = \sum_{k,l} \alpha_{kl}^r \cdot q^k p^l$$

where α_{kl}^r are polynomials in Z^i_j. From equation (3.8) we obtain

$$[Y, p_j + \epsilon_j q_{-j} p_0] = 0 \tag{3.8'}$$

and from (3.9) it follows that

$$[q_k \cdot p_k, \gamma_r] = (2r + 1)\gamma_r \tag{3.9'}$$

A comparision with (3.3) and (3.4′) shows that we have the same problem as in the proof of (ii). The only difference is the factor $(2r + 1)$ in the right-hand side of (3.9′) instead of $2r$ in (3.4′). Using the same argument as in the proof of (ii) one finds $Y = 0$ since with the factor $(2r + 1)$ instead of $2r$ the necessary condition for $\alpha_{kl}^r \neq 0$ reads $k - l = 2r + 1$, which cannot be fulfilled even for α_{00}^0. So we get $\vartheta(q_i) = q_i$ and it follows then immediately from (3.7) and (3.8) that $\vartheta(p_i) = p_i$. Therefore (3.9) turns into

$$[\vartheta(q_0) - q_0] p_0 = (\alpha - \tilde{\alpha}) \cdot \mathbb{1} \tag{3.10}$$

which in the Weyl algebra, where negative powers in p_0 do not occur, can be fulfilled only if $\tilde{\alpha} = \alpha$. This proves (iii). From (3.10) we get further that $\vartheta(q_0) = q_0$ because of the absence of nonzero zero divisors in the Weyl algebra. So we assume $\tilde{\alpha} = \alpha$ and show (iv) as follows. Since the canonical pairs from the subalgebra W_{2m} commute with the remaining $2n - 1$ canonical pairs in $W_{2(2n-1+m)}$, $\vartheta(a)$ for $a \in W_{2m}$ cannot depend on these remaining $2n - 1$ pairs because of Assertion 1. Thus ϑ restricted to W_{2m} must be an endomorphism $\hat{\vartheta}$ of W_{2m}. Therefore the relations (3.6) $\vartheta(\tilde{X}_\beta^\alpha) = X_\beta^\alpha$ taken for $X^i{}_j$ imply

$$\hat{\vartheta}(\tilde{Z}_j^i) = Z_j^i \tag{3.11}$$

On the other hand if $\tilde{Z}^i{}_j$ and $Z^i{}_j$ are related, i.e., there exists an endomorphism $\hat{\vartheta}$ of W_{2m} such that (3.11) holds, then the identical extension of $\hat{\vartheta}$ to an endomorphism ϑ of $W_{2(2n-1+m)}$ yields (3.6), i.e., X_β^α and $\tilde{X}_\beta^\alpha$ are related; so the proof is completed.

The obviously inducing character of Theorem 1 gives rise to the construction of d-parameter sets of canonical realizations of $sp(2n, R)$. For this purpose let us define "signatures" as the $(n + l)$-tuples $(d; 0, \ldots, 0, \alpha_{n-d+1}, \ldots, \alpha_n)$, where $d = 1, 2, \ldots, n$ and α_i are real numbers

Theorem 2. To every signature $(d; 0, \ldots, 0, \alpha_{n-d+1}, \ldots, \alpha_n)$ there corresponds a canonical realization of $sp(2n, R)$ in $W_{2N(d)}$, $N(d) = d(2n - d)$ defined as follows: (a) $(1; 0, \ldots, 0, \alpha_n)$ denotes the realization (3.1) of $sp(2n, R)$ where $\alpha = \alpha_n$ and $Z^i_j = 0$. (b) $(d; 0, \ldots, 0, \alpha_{n-d+1}, \ldots, \alpha_n)$ $d > l$ denotes the realization (3.1), where $\alpha = \alpha_n$ and the realization of $sp(2n - 2, R)$ has the signature $(d; 0, \ldots, 0, \alpha_{n-d+1}, \ldots, \alpha_{n-1})$. For these realizations it holds that

(i) The realizations are skew-Hermitian.
(ii) The realizations are Schur realizations
(iii) Two realizations are related if and only if their signatures are the same

Theorem 2 follows immediately from Theorem 1. The number of canonical pairs is

$$\sum_{k=n-d+1}^{n} (2k - 1) = d(2n - d)$$

Remark. For property (iii) it is of course assumed that $W_{2N(d)}$ is naturally embedded in $W_{2N(d')}$ if $d < d'$.

4. *Concluding Remarks*

A. If we consider the generators X^{α}_{β} given by (3.1) as the basis of a linear space over $\mathbb{C}$, i.e. if we replace $sp(2n, R)$ by its complexification $sp(2n, \mathbb{C})$, then all assertions of Theorems 1 and 2 are also true with exception of skew-Hermiticity, which has no sense for complex Lie algebras. The parameters α_i, $i = n - d + 1, \ldots, n$, then can be taken from $\mathbb{C}$ since the restriction to real parameters was caused only by skew-Hermiticity and other parts of the proof do not depend on this restriction. So, together with the results from Havlíček and Exner (1975a) and Havlíček and Lassner (1976) series of canonical realizations with the same properties described here for $sp(2n, C)$ are given for all the four fundamental series A_n, B_n, C_n, D_n, of the Cartan classification of complex simple Lie algebras.

B. A very well-known method of getting, for an arbitrary Lie algebra L with a basis $x^1, \ldots, x^n$, a canonical realization that is bilinear in q_i and p_i starts with a $N \times N$ matrix representation $X^{\alpha} = (X^{\alpha}_{ij})$ of the generators x^{α} of L and uses the formula

$$\tau(x^{\alpha}) \equiv \tilde{X}^{\alpha} = \sum_{i,j=1}^{N} q_i X^{\alpha}_{ij} p_j \tag{4.1}$$

Formula (4.1) already used by Schwinger in 1952 for the Lie algebra $su(2)$ was generalized and used for Lie algebras of noncompact groups [$U(6, 6)$] in hadron classification by Dothan et al. (1965) with the help of Hermann and Feynman (see related remarks in Dothan et al., 1965. As pointed out by Cordero and Ghirardi (1972) in the review article, some properties of formula (4.1) restrict strongly its generality. So, the number N of canonical pairs depends on the existence of a NxN matrix representation of the Lie algebra L. Further, the realizations by bilinear expressions in q_i and p_i give only a small subset of all possible canonical realizations and the minimal realizations are usually not of this type.

It is possible to generalize formula (4.1): The commutation relations among $\tau(x^\alpha)$ will be conserved if we substitute q_ip_j by E_{ij} satisfying the commutation relations of the Lie algebra $gl(N)$, i.e., if

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj} \tag{4.2}$$

then

$$\tau(x^\alpha) = \sum_{i,j=1}^{N} X_{ij}^\alpha E_{ij} \tag{4.3}$$

fulfil the commutation relations of the algebra L. Whereas in the literature for E_{ij} there were often used q_ip_j or, because of skew-Hermiticity $q_ip_j + \frac{1}{2}\delta_{ij}$, we shall stress the possibility of taking other realizations of $gl(N)$ for E_{ij}. So we can use, e.g., the canonical realizations of $gl(N)$ given in Havlíček and Lassner (1976) and among them the one-parameter set of minimal realizations in $W_{2(N-1)}$ [see footnote 2] which reduce the number of canonical pairs in comparison to those used in (4.1).

The canonical realizations of $sp(2n, R)$ (3.1) with $Z_j^i = 0$ can be got by (4.3) using just the mentioned minimal one-parameter set of realizations of $gl(2n, R)$ and the matrix representation (2.3) of the generators of $sp(2n, R)$.

As the matrix representation (2.3) of $sp(2n, R)$ is real and the realization of $gl(2n, R)$ is skew-Hermitian, the realization of $sp(2n, R)$ defined by (4.3) is skew-Hermitian, too.

C. To take a representation of L by real matrices is not, of course, the only possibility by means of (4.3) to get a skew-Hermitian realization of L. The suitable choice of a representation of L by NxN-dimensional complex matrices in combination with a suitable non-skew-Hermitian realization of $gl(N, R)$ can lead also to a skew-Hermitian realization of L. If we are interested, at the same time, in the realizations with the smallest number of canonical pairs, we have to use a matrix representation with the smallest dimension N. For example, taking the fundamental n-dimensional ($2n$-dimensional) skew-Hermitian representation of $L = su(n)$ [$=sp(2n)$] [see footnote 3] and the realization of $gl(n, R)$ [$gl(2n, R)$] given by

[2] $E_{\mu\nu} = q_\mu p_\nu + \frac{1}{2}\delta_{\mu\nu}$, $E_{N\mu} = -p_\mu$, $E_{\mu N} = q_\mu(q_\nu p_\nu + \frac{1}{2}N - i\alpha)$ $E_{NN} = -q_\nu p_\nu - \frac{1}{2}(N-1) + i\alpha$, $\mu, \nu = 1, \ldots, N-1$, $\alpha \in \mathbb{R}$ [eq. (11) in Havlíček and Lassner (1975)]. These realizations are skew-Hermitian and they are Schur realizations.

[3] The compact form of the algebra C_n from the Cartan classification.

$$E_{ij} = \frac{1}{2}(p_i p_j + q_i p_j - q_j p_i - q_i q_j)$$

for which $E_{ij}^+ = E_{ji}$ holds, we obtain a skew-Hermitian realization of $su(n)$ [$sp(2n)$] in W_{2n} ($W_{2 \cdot 2n}$).

These realizations are *minimal skew-Hermitian* realizations of the Lie algebras $su(n)$ and $sp(2n)$, i.e., realizations with the smallest number of canonical pairs among all skew-Hermitian realizations. The following considerations show that in $W_{2(n-1)}$ ($W_{2(2n-1)}$) no skew-Hermitian realization of $su(n)$ [$sp(2n)$] exists. Joseph (1972, Theorem 4.4) has showed that no skew-Hermitian nontrivial realization of a compact Lie algebra is a Schur realization. It was shown, however, that all realizations of the Lie algebra A_{n-1} in $W_{2(n-1)}$ are Schur realizations (Joseph, 1972; Simoni and Zaccaria, 1969). Consequently, the same assertion takes place for any real form of A_{n-1} including $su(n)$ and therefore a skew-Hermitian realization of $su(n)$ in $W_{2(n-1)}$ does not exist. The same assertion is valid for $sp(2n)$ in $W_{2(2n-1)}$ because, as we mentioned in the Introduction, any realization of $sp(2n, R)$ in $W_{2(2n-1)}$ is a Schur realization and $sp(2n)$ is another real form of the common complexification $sp(2n, C)$.

Since, it was proved in Havlíček and Exner (1975b) that minimal skew-Hermitian canonical realizations of the Lie algebras $o(n)$ exist in $W_{2(n-1)}$, the problem of the existence of these realizations is solved completely for all compact classical Lie algebras.

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